ADJUNCTIONS BETWEEN DEFAULT FRAMEWORKS

PART 1

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ABSTRACT

We explore the relationship between preference-based and distance-based semantics for defaults. This is done by defining several categories for default semantics. Some of these are preference-based, some are distance-based, and some are mixtures of the two paradigms. We exhibit the relationships between the categories by defining maps between them, which we show to be adjunctions.

1. Introduction

A *default* is a piece of information which expresses a generality for which it is known that there may be exceptions. The need to be able to express defaults in specifications is well-established[1, 10, 13]. For example, using defaults we may specify the intended behaviour of a system in such a way that this behaviour may be overridden when some exceptional circumstance arises. Using defaults makes specifications more readable, more modular, and more reusable[10].

Typically, a specification will consist of two parts: some facts and some default information. The facts are the statements that must hold in the specification, and the defaults are statements which should hold provided there is no contrary evidence. The defaults may also be ordered by priority[2, 5, 9]. In this paper we will not discuss the precise structure of the default information (for example, whether it is prioritised or not).

Our aim is to explore the relationship between several model-theoretic frameworks for the semantics of defaults. Such frameworks adopt the standpoint that, given certain facts and defaults, we must look at those models of the facts which come closest to satisfying the default information. The point is that it is not in general possible for models of the facts to satisfy the defaults. The criteria for choosing among the models of the facts is that they satisfy the defaults as much as possible.

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We may broadly distinguish between two approaches in model-theoretic semantics for defaults: the preference-based approach and a distance-based approach. The preference-based approach[7, 8, 12, 1] employs a *preference relation* which orders interpretation structures, or perhaps sets of interpretation structures, according to how well they satisfy the default information. By convention, those interpretations which are *lower* in the ordering are the ones which better satisfy the defaults.

The distance-based approach[13, 14, 11] stipulates a measure of distance between two interpretations. This is independent of the default information, and is used in conjunction with the default information to find the models of the theory which are closest to the models of the default information.

In this paper we adopt the categorical standpoint whose slogan is: "know your objects via their morphisms"[6]. The meaning of this slogan is that a mathematical object can be understood properly only by also understanding the relationship between such objects. Thus, for instance for the preference-based approach, as well as studying preference relations, we should also study how preference relations are related to each other. There are several possible relationships between preference relations which we could define — in categorical language, we say that there are several notions of morphism — but an intuitive possibility is to focus on their granularity of defeasibility.

Suppose we have some facts F and some defaults D. Default logics work by reasoning classically with F together with some part of D which is consistent with F. Where they might differ is in the extent to which they decompose D to find components of it which are consistent with F. One default logic may have a very coarse granularity, just checking whether D is consistent as a whole with F. Another logic may be much finer, splitting D in many ways to find parts of it which are consistent with F. The two logics described differ in their granularity of defeasibility.

In terms of preference relations, we will see that one preference relation is coarser than another one if the first is a subrelation of the second. Thus, our notion of morphism in the category $\mathcal{P}$ of preference relations is simply inclusion.

In a similar way we will describe three other categories, which are called $\mathcal{D}$, $\mathcal{D}^-$, and $\mathcal{DP}$. The objects in each category share a mechanism for handling defaults, such as the preference relation mechanism of $\mathcal{P}$. The morphisms express whether one such object is coarser (or finer) than another.

Therefore, more specifically, our aim is to be able to translate between these categories of default logics in such a way that we preserve the granularities. Concretely, we will seek to establish adjunctions between the categories. An adjunction between two categories is a pair of maps going in opposite directions between the categories, such that the round trip from either category back to itself via the composition of the maps returns an object which bears the sort of relationship to the original object that is specified by the morphism. This will be stated formally later in the paper (definition 1.2.3). The notion of adjunction is a generalisation of the notion of inverse.

No prior exposure to category theory is needed to understand this paper. Indeed, for most of the paper we will require only a special case of the notion of category, called a *pre-order category*, in which there is at most one morphism between any two objects. This is much simpler than the general case, and the notions of functor and adjunction which we will require (and define) are correspondingly simpler.

The paper is structured as follows. In the remainder of this section, we review some preliminaries. Section 2 is the main section, in which we introduce the two most important examples of default frameworks, namely $\mathcal{P}$ and $\mathcal{D}$ (sections 2.1 and 2.2). Their relationship is explored in section 2.3 by giving adjunctions between these and other categories. A diagram illustrating the various adjunctions is given in figure 1. In section 3 we make some remarks about the application of these results to the idea of default institutions. Finally, we conclude with section 4.

This paper is the first part of some work that will be continued in a future paper.

1.1. Assumptions and notation

Throughout this paper, we will assume a collection $\mathcal{M}$ of interpretation structures. We will not refer explicitly to any logical language; instead, we will treat a sentence or a theory simply as the set of its models; that is, as a subset of $\mathcal{M}$.

If $<$ is a relation on the set $\mathcal{X}$ and $Y \subseteq \mathcal{X}$, then $y \in Y$ is said to be *$<$-minimal in* Y iff $\forall y' \in Y.\ (y' \not< y)$. We define $\mathrm{Min}_<(Y) = \{y \in Y \mid y \text{ is } <\text{-minimal in } Y\}$. We use $\subseteq$ to mean subset, and $\subset$ to mean strict subset.

1.2. Pre-order categories

In this section we briefly review the category theory we require. There are many books on category theory to which the reader may refer for a fuller picture, but we particularly recommend the Chapter 2 of Crole[3] for its readability and because it deals explicitly with the special case of pre-order categories which we require for this paper.

1.2.1 Definition. A *pre-order category* $(\mathcal{C}, \rightarrow)$ is simply a pre-order, that is, a set $\mathcal{C}$ equipped with a reflexive and transitive relation $\rightarrow$. When $A \rightarrow B$ $(A, B \in \mathcal{C})$ we say that there is a morphism from A to B. If $A \rightarrow B$ and $B \rightarrow A$, we write $A \leftrightarrow B$.

1.2.2 Definition. Let $(\mathcal{C}, \rightarrow_\mathcal{C})$ and $(\mathcal{D}, \rightarrow_\mathcal{D})$ be pre-order categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a monotonic map from $\mathcal{C}$ to $\mathcal{D}$, that is, a map such that if $A \rightarrow_\mathcal{C} B$ is a morphism in $\mathcal{C}$ then there is a morphism $F(A) \rightarrow_\mathcal{D} F(B)$ in $\mathcal{D}$. If F is such a functor we write $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

1.2.3 Definition. Let $(\mathcal{C}, \rightarrow_\mathcal{C})$ and $(\mathcal{D}, \rightarrow_\mathcal{D})$ be pre-order categories, and L and R be functors from $\mathcal{C}$ to $\mathcal{D}$ and $\mathcal{D}$ to $\mathcal{C}$ respectively, a situation we may write as

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}\,.$$

Then L and R are said to form an *adjunction*, with L as the *left adjoint* and R as the *right adjoint*, if

- For any $A \in \mathcal{C}$, there is a morphism $A \rightarrow_\mathcal{C} R(L(A))$ in $\mathcal{C}$, and
- For any $B \in \mathcal{D}$, there is a morphism $L(R(B)) \rightarrow_\mathcal{D} B$ in $\mathcal{D}$.

We write the fact that L is left-adjoint to R as $L \dashv R$.

1.2.4 Proposition. The following facts hold about adjunctions. Proofs may be found in Crole[3].

1. $L \dashv R$ iff for each $A \in \mathcal{C}$ and $B \in \mathcal{D}$, $(L(A) \rightarrow_\mathcal{D} B$ iff $A \rightarrow_\mathcal{C} R(B))$.
2. When adjoints exist they are unique up to isomorphism. If $\mathcal{C} \xrightarrow{L} \mathcal{D}$ is a functor and if $L \dashv R_1$ and $L \dashv R_2$ then, for all $B \in \mathcal{D}$, $R_1(B) \leftrightarrow_D R_2(B)$. Similarly, left adjoints are also unique up to isomorphism.
3. Adjunctions compose to give other adjunctions. Suppose

   $$\mathcal{C} \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathcal{D} \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathcal{E}$$

   are functors and $L_1 \dashv R_1$ and $L_2 \dashv R_2$. Then

   $$\mathcal{C} \underset{R_1 \circ R_2}{\overset{L_2 \circ L_1}{\rightleftarrows}} \mathcal{E}$$

   are functors and $L_2 \circ L_1 \dashv R_1 \circ R_2$.

There is a special case of adjunction known as the co-reflection, which has stronger conditions.

1.2.5 Definition. An adjunction $L \dashv R$ as described in 1.2.3 between $\mathcal{C}$ and $\mathcal{D}$ is a *co-reflection* if, for each $A \in \mathcal{C}$, we have $R(L(A)) \leftrightarrow A$.

Essentially, this means that $\mathcal{C}$ is isomorphic to a subcategory of $\mathcal{D}$.

2. Model-based frameworks for defaults

2.1. The preference-relation framework $\mathcal{P}$

The preference-relation framework $\mathcal{P}$ is based on the idea that interpretations are ordered according to how well they satisfy the defaults; compare with the standard works[7, 8, 12, 1] on preferential models. Suppose some default information is given. We assume that, however it is presented syntactically, there is some machinery which we can use to reduce it to a set M of interpretations. The set M is the set of 'ideal' or 'most normal' models as far as the defaults are concerned.

The objects inside $\mathcal{P}$ are thus preference relations indexed by subsets of $\mathcal{M}$. If $M \subset \mathcal{M}$ then $\prec_M$ is the relation on $\mathcal{M} \times \mathcal{M}$ which orders the elements in $\mathcal{M}$ according to how close they are to being inside M. The expression $x \prec_M y$ means that x is closer to being a member of M than y is.

2.1.1 Definition. An object in the category $\mathcal{P}$ of preference relations is a relation $\prec$ in $\mathcal{M} \times \mathcal{P}(\mathcal{M}) \times \mathcal{M}$, such that

1. transitivity: $x \prec_M y \prec_M z$ implies $x \prec_M z$.
2. base: If $x \in M$ and $y \notin M$ then $x \prec_M y$; and if $x \prec_M y$ then $y \notin M$ and $M \neq \varnothing$.
3. chain: there is no infinite descending chain of the form
$$\ldots \prec_M x_3 \prec_M x_2 \prec_M x_1.$$
4. unions: If $I \neq \varnothing$, then $\forall i \in I.\ x \prec_{M_i} y$ implies $x \prec_{\bigcup_{i\in I} M_i} y$.

A morphism exists between objects $\prec_1$ and $\prec_2$, written $\prec_1 \to \prec_2$ iff $\prec_1 \subseteq \prec_2$ as relations.

The transitivity condition is very intuitive; for if x is closer to M than y is, and that y is in turn closer than z, then x must also be closer than z. The condition called base means that the elements of M are precisely those which are minimal in $\prec_M$; this is proved in the next proposition. The chain condition is there for technical reasons, namely that we will require the ability to find models which are $\prec_M$ minimal within arbitrary subsets of $\mathcal{M}$. Note that the chain condition also implies that $\prec_M$ is irreflexive. The unions condition says that if x is closer than y to each of a family $\{M_i\}_{i\in I}$ of sets, then x is closer than y to the union of those sets.

The chain condition corresponds to the *stopperedness* condition prevalent in the literature[8]. Usually stopperedness is stated as saying that, for any $x \in \mathrm{Mod}(\phi)$ there is a $y < x$ with y minimal in $\mathrm{Mod}(\phi)$. In the setting of this paper, we simplify by treating formulas just as the set of their models. Therefore the condition of stopperedness becomes: for any $M \subseteq \mathcal{M}$ and $x \in M$ there is a $y < x$ with y minimal in M. This is equivalent to the chain condition in the definition above.

2.1.2 Proposition. y is $\prec_M$-minimal iff $y \in M$ or $M = \varnothing$.

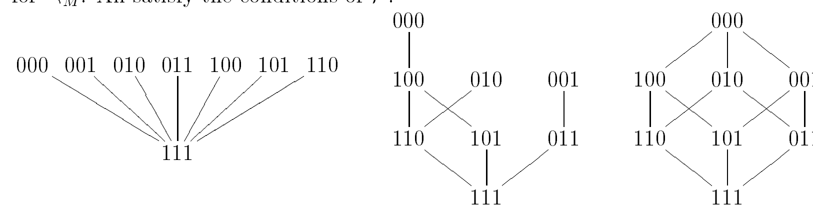
Proof. $\Rightarrow$ If $y \in \mathrm{Min}_{\prec_M}(\mathcal{M})$ and $M \neq \varnothing$, pick any $x \in M$. Then $x \not\prec_M y$ so by the base condition, $y \in M$.

$\Leftarrow$ Suppose $y \notin \mathrm{Min}_{\prec_M}(\mathcal{M})$; we will prove that $y \notin M$ and $M \neq \varnothing$. Take x such that $x \prec_M y$. Then by base, $y \notin M$, and $M \neq \varnothing$. ∎

The notion of morphism in $\mathcal{P}$ is supposed to reflect the granularity of defeasibility. Thus, if $\prec$ is coarser than $\prec'$, we expect that more should be derivable from a set of facts and a set of defaults using $\prec'$ than with $\prec$. This means that there should be fewer minimal models (more information derivable) of N according to $\prec'_M$ than according to $\prec_M$. This is indeed the case.

2.1.3 Proposition. $\prec \to \prec'$ iff $\forall M \subseteq \mathcal{M}, N \subseteq \mathcal{M}.\ \mathrm{Min}_{\prec'_M}(N) \subseteq \mathrm{Min}_{\prec_M}(N)$

2.1.4 Example. Consider the default $p \wedge q \wedge r$, represented by the set of models $M = \{111\}$. (Notation: 001 is the element of $\mathcal{M}$ which assigns false to p and q and true to r, etc.) Then the following three Hasse diagrams represent three possibilities for $\prec_M$. All satisfy the conditions of $\mathcal{P}$.



Moreover, the three possibilities for $\prec_M$ are in ascending order of granularity. All distinctions made by the first are also made in the second, and all those of the second occur also in the third.

2.2. *Distance functions; the category $\mathcal{D}$*

We now turn to a very different way of structuring the set $\mathcal{M}$ in order to equip it to deal with defaults. We will study the idea of stipulating a *distance function* which will control defeasibility. A distance function takes two interpretations and returns a value which is interpreted as measuring how different the two interpretations are. Appealing examples include the following:

2.2.1 Examples (Cf. Schobbens[13, 11]). Let $\mathcal{M}$ be the set of propositional valuations for finitary propositional logic (i.e. the number of proposition symbols is finite). The coarsest distance function is the following one, which simply returns 'zero' or 'unit' distance:

$$d_1(x,y) = \begin{cases} \varnothing & \text{if } x = y \\ \{*\} & \text{otherwise} \end{cases}$$

Another distance function is the following one,

$$d_2(x,y) = \langle \{p \mid x \Vdash p \text{ and } y \nVdash p\}, \{p \mid x \nVdash p \text{ and } y \Vdash p\} \rangle$$

which gives us a pair of sets: the proposition symbols satisfied by x but not by y on the one hand, and those satisfied by y but not x on the other.

A third example is

$$d_3(x,y) = \{p \mid x \Vdash p \text{ and } y \nVdash p\} \cup \{p \mid x \nVdash p \text{ and } y \Vdash p\},$$

which says that the distance between two interpretations is simply the set of proposition symbols on which they differ.

In general, and in the examples above, distance values are not real numbers. They are of course ordered, but not necessarily totally ordered. We will allow any partial order. In the examples above, d_1 and d_3 are ordered by subset, while d_2 is ordered by subset on the components.

2.2.2 Definition. An object $(d,<)$ in the category $\mathcal{D}$ of distance functions is a function d from $\mathcal{M}\times\mathcal{M}$ to a strict partial order $(\Delta,<)$, satisfying

1. base: if $y \neq v$ then $d(x,x) < d(y,v)$; and if $d(x,u) < d(y,v)$ then $y \neq v$.
2. chain: there is no infinite descending chain $\ldots < d(x_3,u_3) < d(x_2,u_2) < d(x_1,u_1)$.

A morphism exists between objects $(d_1,<_1)$ and $(d_2,<_2)$, written $(d_1,<_1) \to (d_2,<_2)$ or sometimes more briefly as $d_1 \to d_2$, iff for all $x,y,u,v \in \mathcal{M}$, $d_1(x,u) <_1 d_1(y,v)$ implies $d_2(x,u) <_2 d_2(y,v)$.

As before, the notion of morphism is intended to capture granularity.

The reader might expect a condition saying that $(\Delta,<)$ contains a bottom element $\perp$ and that $d(x,u)=\perp$ iff $x=u$. We do indeed allow such an arrangement, in the sense that the conditions we impose in 2.2.2 are weaker. However, it turns out that it is more convenient not to stipulate a single minimum value. The conditions of 2.2.2 are strong enough for our purposes; in particular:

2.2.3 Proposition. $d(y,v)$ is minimal in the range of d iff $y=v$. Moreover, in the range of d every minimal is less than every non-minimal.

Proof.

$y=v \Rightarrow$	$\forall x,u.\ d(x,u) \not< d(y,v)$	by second part of base
$\Leftrightarrow$	$d(y,v)$ minimal in the range of d	by definition of minimal
$\Rightarrow$	$y=v$	by first part of base

The second part of the proposition follows immediately from the base condition. ■

2.2.4 Proposition. The three examples in 2.2.1 are in $\mathcal{D}$, with $d_1 \to d_2$ and $d_2 \to d_3$.

To check whether the notion of morphism really does capture granularity, we must define how a distance function is to be used. Given a set M of 'ideal' interpretations according to the defaults and a set N of models of the facts, we wish to find (as we did in the preference relations case) which models in N are closest to being in M.

2.2.5 Definition. Let $M,N \subseteq \mathcal{M}$. The models in N which are closest (according to d) to M are given by

$$C_M(N) = \{x \in N \mid \forall x' \in N.\ \exists y \in M.\ \forall y' \in M.\ d(x',y') \not< d(x,y)\}$$

This says that the closest elements of N to M are those elements x of N for which, given another element x' of N, there is a distance from x to M which is minimal among all distances x' to M.

Intuitively, d_1 being coarser than d_2 means that d_2 will report fewer interpretations in N closest to M (i.e. more information) than d_1. Our notion of morphism must therefore verify the following proposition:

2.2.6 Proposition. $(d_1,<_1) \to (d_2,<_2)$ implies $C^2_M(N) \subseteq C^1_M(N)$.

Proof. Suppose $(d_1,<_1) \to (d_2,<_2)$.

$$\begin{aligned} & x \notin C^1_M(N) \\ \Leftrightarrow\ & x \notin N \vee (\exists x' \in N.\ \forall y \in M.\ \exists y' \in M.\ d_1(x',y') <_1 d_1(x,y)) \\ \Rightarrow\ & x \notin N \vee (\exists x' \in N.\ \forall y \in M.\ \exists y' \in M.\ d_2(x',y') <_2 d_2(x,y)) \quad \text{by hypothesis} \\ \Leftrightarrow\ & x \notin C^2_M(N) \end{aligned}$$

■

2.3. The relationship between $\mathcal{D}$ and $\mathcal{P}$

Our aim at the outset of this work was to establish an adjunction between $\mathcal{P}$ and $\mathcal{D}$. It is still an open question whether this is possible. In this paper, we will content ourselves by providing adjunctions between $\mathcal{P}$ and $\mathcal{D}$ via several other categories which will introduce in this section.

To motivate this further, we'll now describe a functor from $\mathcal{D}$ to $\mathcal{P}$ which has several good properties. Think of this as a way of translating a distance function in $\mathcal{D}$ into a preference relation in $\mathcal{P}$ which is somehow equivalent to it. Given a distance function d, we'll define the preference relation $\prec$ as follows:

$$x \prec_M y \overset{\text{def}}{\Longleftrightarrow} M \neq \varnothing \wedge \forall v \in M.\ \exists u \in M.\ d(x,u) < d(y,v). \qquad (*)$$

To check whether this really is a functor, it is necessary to verify that the $\prec$ thus defined satisfies the conditions of $\mathcal{P}$ given in 2.1.1, assuming that d satisfies the conditions of $\mathcal{D}$ in 2.2.2. It is also necessary to verify that if $d_1 \to d_2$ and d_1,d_2 are transformed to $\prec_1,\prec_2$, then $\prec_1 \to \prec_2$. All these facts are true; we'll verify them later in this section.

The intuition behind the definition is simple. It says that x is closer to the set M than y is if ($M \neq \varnothing$ and) for every distance y to M there is a distance x to M which is smaller than it. The proviso about the empty set is just to express the fact that no point is closer than some other point to being in the empty set.

This way of transforming elements of $\mathcal{D}$ to elements of $\mathcal{P}$ has the property that if we focus on any subsets M,N of $\mathcal{M}$ and ask which elements of N are closest to M, then d and $\prec$ gives us the same answer.

2.3.1 Proposition. Let d and $\prec$ be related by $(*)$ above, and $M,N \subseteq \mathcal{M}$. Then

$$C_M(N) = \mathrm{Min}_{\prec_M}(N).$$

Proof.

$$\begin{aligned} y \in \mathrm{Min}_{\prec_M}(N) \Leftrightarrow\ & y \in N \wedge \forall x \in N.\ x \not\prec_M y \\ \Leftrightarrow\ & y \in N \wedge \forall x \in N.\ \exists v \in M.\ \forall u \in M.\ d(x,u) \not< d(y,v) \\ \Leftrightarrow\ & y \in C_M(N) \end{aligned}$$

■

The open question alluded to at the beginning of this section can now be cast as follows: does this functor have an adjoint? In this paper, we go part of the way towards answering it. We define the intermediate categories $\mathcal{D}^-$ and $\mathcal{DP}$. We give

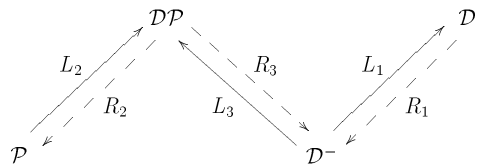


Fig. 1. The following diagram illustrates all the categories of default logics and the functors between them. Each L functor is left-adjoint to the corresponding R functor. Moreover, each adjunction $L \dashv R$ is a co-reflection. This means that the composition $R \circ L$ is an isomorphism. Thus, for example, if you start with $\prec$ in $\mathcal{P}$ and map it via L_2 to $\mathcal{DP}$ and then back again via R_2, you get back to the same relation $\prec$.

several adjunctions between them and $\mathcal{D}$ and $\mathcal{P}$, which are summarised in the diagram in figure 1. The composition of the functors R_1, L_3 and R_2 in the figure is the functor described by $(*)$ above.

The two categories which mediate between $\mathcal{D}$ and $\mathcal{P}$ are intuitive and help clarify the relationship. The first category generalises $\mathcal{D}$ slightly. Instead of positing concrete distance values and then imposing an ordering on them, as we did in $\mathcal{D}$, we simply order pairs of interpretations.

2.3.2 Definition. An object in the category $\mathcal{D}^-$ is a relation $<$ in $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M}$ such that

1. transitivity: $x, u < y, v < z, w$ implies $x, u < z, w$.
2. base: if $y \neq v$ then $x, x < y, v$; and if $x, u < y, v$ then $y \neq v$.
3. chain: there is no infinite descending chain $\ldots < x_3, u_3 < x_2, u_2 < x_1, u_1$.

A morphism exists between objects $<_1$ and $<_2$ iff $<_1 \subseteq <_2$ as relations.

Thus, $\mathcal{D}^-$ abstracts away from the particular values of $d(x, y)$. In $\mathcal{D}^-$, we can only make comparisons of the form $d(x, u) < d(y, v)$, which we write simply as $x, u < y, v$; we cannot look at the actual values of $d(x, u)$. Notice that the chain condition in $\mathcal{D}^-$ implies irreflexivity.

The relationship between $\mathcal{D}$ and $\mathcal{D}^-$ will be established by giving functors between them which form an adjunction.

2.3.3 Definition/proposition. The functors $\mathcal{D}^- \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathcal{D}$ are defined by

- $L_1(<)\ (x, u) = \{(z, w) \mid z, w \leqslant x, u\}$, ordered by proper inclusion.
- $x, u\ R_1(d)\ y, v \Leftrightarrow d(x, u) < d(y, v)$.

This is a 'definition/proposition' because it defines the functors; however, it is necessary to check that they are indeed functors.

The intuition for R_1 is simple: it abstracts away from the distance values in $\mathcal{D}$. L_1, on the other hand, introduces some concrete distance values which are simply down-sets in $\mathcal{D}^-$.

Proof. We verify that

(a) $L_1(<)$ satisfies the conditions of $\mathcal{D}$.

(b) $R_1(d, <)$ satisfies the conditions of $\mathcal{D}^-$.

(c) L_1 preserves morphisms, i.e. if $<_1 \to <_2$ then $L_1(<_1) \to L_1(<_2)$.

(d) R_1 preserves morphisms, i.e. if $(d_1, <_1) \to (d_2, <_2)$ then $R_1(d_1, <_1) \to R_1(d_2, <_2)$.

The details of these verifications may be found in the Appendix. ■

2.3.4 Proposition. $L_1 \dashv R_1$; moreover, this adjunction is a co-reflection[†].

Proof. We verify the following.

1. $L_1(R_1((d, <))) \to (d, <)$. Let $(d', <') = L_1(R_1((d, <)))$.

$$
\begin{aligned}
& d'(x,u) <' d'(y,v) \\
& \quad \Rightarrow \{(z,w) \mid z,w = x,u \vee z,w R_1(d,<) x,u\} \\
& \qquad\quad \subseteq \{(z,w) \mid z,w = y,v \vee z,w R_1(d,<) y,v\} \wedge x,u \neq y,v \\
& \quad \Rightarrow \forall z,w.\ (z,w = x,u \vee d(z,w) < d(x,u) \Rightarrow \\
& \qquad\quad z,w = y,v \vee d(z,w) < d(y,v)) \wedge x,u \neq y,v \\
& \quad \Rightarrow (x,u = y,v \vee d(x,u) < d(y,v)) \wedge x,u \neq y,v \qquad \text{by putting } z,w = x,u \\
& \quad \Rightarrow d(x,u) < d(y,v).
\end{aligned}
$$

2. $R_1(L_1(<)) \leftrightarrow <$.

$$
\begin{aligned}
& x,u R_1(L_1(<)) y,v \\
& \quad \Leftrightarrow L_1(<)\ (x,u) \subset L_1(<)\ (y,v) \\
& \quad \Leftrightarrow \forall z,w.\ (z,w \leqslant x,u \Rightarrow z,w \leqslant y,v) \wedge \\
& \qquad\quad \exists z,w.\ (z,w \nleqslant x,u \wedge z,w \leqslant y,v) \\
& \quad \Leftrightarrow x,u < y,v. \ ■
\end{aligned}
$$

[†]Reinhold Heckmann has pointed out that this co-reflection is actually an equivalence. In other words, L_1 and R_1 are not merely adjoints, but are inverses (up to isomorphism in $\mathcal{D}$). This means we need never work with concrete distance values, but can always abstract away from them as we do in $\mathcal{D}^-$. It also means that the diagram in figure 1 should be drawn with $\mathcal{D}^-$ and $\mathcal{D}$ at the same level. There is no need to label the functors between them with names suggesting left and right, since we have $L_1 \dashv R_1$ *and* $R_1 \dashv L_1$.

Technically, the proof of equivalence can be made as follows. Condition (d) in the proof of 2.3.3 can be strengthened to $(d_1, <_1) \to (d_2, <_2)$ iff $R_1(d_1, <_1) \to R_1(d_2, <_2)$. From 2.3.4 we have $R_1(L_1(R_1(d))) = R(d)$, which by the strengthening of (d) yields $L_1(R_1(d)) \leftrightarrow d$.

Now our task is to establish a relationship between $\mathcal{D}^-$ and $\mathcal{P}$, i.e. between judgements of the type $x, u < y, v$ and those of the type $x \prec_M y$. The first judgement says that x is closer to u than y is to v, while the second says that x is closer to M than y is to M. The next intermediating category is designed to combine these two types of judgements. We will define a category $\mathcal{DP}$ in which $x, M \sqsubset y, N$ means that x is closer to M than y is to N. Thus, $\mathcal{D}^-$ is the special case in which M, N are singletons, and $\mathcal{P}$ is the special case in which $M = N$.

2.3.5 Definition. An object in the category $\mathcal{DP}$ is a relation $\sqsubset$ in $\mathcal{M} \times \mathcal{P}(\mathcal{M}) \times \mathcal{M} \times \mathcal{P}(\mathcal{M})$ such that

1. *weak* transitivity: $x, M \sqsubset y, M \sqsubset z, M$ implies $x, M \sqsubset z, M$; and $x, \{u\} \sqsubset y, \{v\} \sqsubset z, \{w\}$ implies $x, \{u\} \sqsubset z, \{w\}$.
2. base: if $x \in M$ and $y \notin N$ then $x, M \sqsubset y, N$; and if $x, M \sqsubset y, N$ then $y \notin N$ and $M \neq \varnothing$.
3. chain: there are no infinite descending chains of the form $\ldots \sqsubset x_3, M_3 \sqsubset x_2, M_2 \sqsubset x_1, M_1$.
4. unions: if $I \neq \varnothing$ then $\forall i \in I.\ x, M_i \sqsubset y, N_i$ implies $x, \bigcup_{i \in I} M_i \sqsubset y, \bigcup_{i \in I} N_i$.
5. monotonicity: $x, M \sqsubset y, N$ and $M \subseteq M'$ and $N' \subseteq N$ implies $x, M' \sqsubset y, N'$.

As before, a morphism exists between objects $\sqsubset_1$ and $\sqsubset_2$ iff $\sqsubset_1 \subseteq \sqsubset_2$ as relations.

This category allows us to compare distances from interpretations to *sets* of interpretations. Thus, $x, M \sqsubset y, N$ means that x is nearer to the set M than y is to the set N; think of this as saying that x is more nearly a member of M than y is of N. Whereas $\mathcal{D}^-$ tells us how to order distances between points, and $\mathcal{P}$ tells us how to order distances from points to a fixed set, $\mathcal{DP}$ does both these jobs at once by telling us how to order distances between points and sets of points. To put this in another way, it is clear how to extract from an object $\sqsubset$ of $\mathcal{DP}$ either an object like those of $\mathcal{D}^-$ or an object like those of $\mathcal{P}$. This is the intuition behind the forgetful functors, which we'll see shortly.

A particular point of interest is that the weak transitivity condition does not guarantee full transitivity of the form

$$x, M \sqsubset y, N \sqsubset z, K \text{ implies } x, M \sqsubset z, K.$$

This is guaranteed to hold only if $M = N = K$ or M, N, K are all singletons. We may think of this as saying that a particular set M imposes an ordering on points x, y, given by $x \prec_M y$ in $\mathcal{P}$ and $x, M \sqsubset y, M$ in $\mathcal{DP}$. This ordering is transitive. But $\mathcal{DP}$ also allows comparison of the orderings given by two different sets M and N. Transitivity of such comparisons is not guaranteed.

The conditions of base and chain resemble those we have seen before in $\mathcal{P}$ and $\mathcal{D}^-$. As before, the chain condition implies irreflexivity. Unions is the natural counterpart of unions in $\mathcal{DP}$, while monotonicity tells us that if you grow M, then the distance x, M can only shrink.

The relationship between $\mathcal{P}$ and $\mathcal{D}^-$ is established by establishing a co-reflection between each of them and $\mathcal{DP}$.

2.3.6 Definition/proposition. The functors $\mathcal{P} \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathcal{DP} \underset{L_3}{\overset{R_3}{\rightleftarrows}} \mathcal{D}^-$ are defined by

- $x, M\ L_2(\prec)\ y, N \Leftrightarrow (\exists K.\ N \subseteq K \subseteq M \wedge x \prec_K y) \vee (x \in M \wedge y \notin N)$.
- $x\ R_2(\sqsubset)_M\ y \Leftrightarrow x, M \sqsubset y, M$.
- $x, M\ L_3(<)\ y, N \Leftrightarrow M \neq \varnothing \wedge \forall v \in N.\ \exists u \in M.\ x, u < y, v$.
- $x, u\ R_3(\sqsubset)\ y, v \Leftrightarrow x, \{u\} \sqsubset y, \{v\}$.

To understand L_2, notice that we start with the ability to order the distances of points from a fixed set ($\mathcal{P}$), and we want to order distances from arbitrary sets ($\mathcal{DP}$). The strategy of this functor is as follows. If it happens that the arbitrary sets are related by inclusion ($N \subseteq M$), then we can use the information we have in $\mathcal{P}$ by looking at intermediate sets (the K). Otherwise, we can only tell in rather degenerate circumstances, namely those of $\mathcal{DP}$-base.

L_3 uses the uper power-ordering. We say that the distance x, M is less than the distance y, N if every distance y to v ($v \in N$) is beaten by some distance x to u ($u \in M$).

On the other hand, the maps R_2 and R_3 simply extract the special case which $\mathcal{P}$ and $\mathcal{D}^-$ are of $\mathcal{DP}$.

Proof. We verify that
(a) $L_2(\prec)$ satisfies the conditions of $\mathcal{DP}$.
(b) $R_2(\sqsubset)$ satisfies the conditions of $\mathcal{P}$.
(c) $L_3(<)$ satisfies the conditions of $\mathcal{DP}$.
(d) $R_3(\sqsubset)$ satisfies the conditions of $\mathcal{D}^-$.
(e) L_2 preserves morphisms, i.e. if $\prec_1 \to \prec_2$ then $L_2(\prec_1) \to L_2(\prec_2)$.
(f) R_2 preserves morphisms, i.e. if $\sqsubset_1 \to \sqsubset_2$ then $R_2(\sqsubset_1) \to R_2(\sqsubset_2)$.
(g) L_3 preserves morphisms, i.e. if $<_1 \to <_2$ then $L_3(<_1) \to L_3(<_2)$.
(h) R_3 preserves morphisms, i.e. if $\sqsubset_1 \to \sqsubset_2$ then $R_3(\sqsubset_1) \to R_3(\sqsubset_2)$.
The details of this verification are in the Appendix. ■

2.3.7 Proposition. $L_2 \dashv R_2$ and $L_3 \dashv R_3$. Moreover, both these adjunctions are co-reflections.

The proof of this proposition is given in the appendix.

3. Default institutions

An *institution*[4] is a presentation of a logic woth emphasis on the possibility of different signatures (vocabularies). Institutions are useful for structuring specifications. Different components of a structured specification use different signatures. We recall the definition of institution:

3.0.8 Definition. An *institution* $\langle \mathrm{Sig}, \mathrm{Sen}, \mathrm{Int}, \Vdash \rangle$ consists of a category Sig of *signatures*, together with the two functors Sen from Sig to Set, and Int from Sig to $\mathrm{Set}^{\mathrm{op}}$. Given a signature Σ, $\mathrm{Sen}(\Sigma)$ and $\mathrm{Int}(\Sigma)$ are the sets of *sentences over* Σ and *interpretations over* Σ respectively. $\Vdash$ is a Σ-indexed family of satisfaction relations; $\Vdash_\Sigma$ is a relation in $\mathrm{Int}(\Sigma) \times \mathrm{Sen}(\Sigma)$.

3.0.9 Definition. A morphism h between institutions $\langle \mathrm{Sig}, \mathrm{Sen}, \mathrm{Int}, \Vdash \rangle$ and $\langle \mathrm{Sig}', \mathrm{Sen}', \mathrm{Int}', \Vdash' \rangle$ is a functor $h : \mathrm{Sig} \to \mathrm{Sig}'$, a natural transformation $h_\alpha : h; \mathrm{Sen}' \Rightarrow \mathrm{Sen}$, and a natural transformation $h_\beta : \mathrm{Int} \Rightarrow h; \mathrm{Int}'$, such that the following *satisfaction condition* holds:

$$x \Vdash_\Sigma h_{\alpha_\Sigma}(\phi') \text{ iff } h_{\beta_\Sigma}(x) \Vdash'_{h(\Sigma)} \phi',$$

for each $x \in \mathrm{Int}(\Sigma)$ and $\phi' \in \mathrm{Sen}'(h(\Sigma))$.

Our categories $\mathcal{P}$, $\mathcal{D}$ etc. can be viewed as candidates for the machinery which must be added to an institution in order to equip it for handling defaults. Thus, definitions 2.1.1 and 3.0.9 can be combined to give:

3.0.10 Definition. Let I be the institution $\langle \mathrm{Sig}, \mathrm{Sen}, \mathrm{Int}, \Vdash \rangle$. Then $\mathcal{P}(I)$ is the preference default institution based on I given by I together with a $\Sigma \times \mathrm{Sen}(\Sigma)$-indexed family of preference relations; $\prec_{\Sigma,\phi}$ is a relation in $\mathrm{Int}(\Sigma) \times \mathrm{Int}(\Sigma)$ satisfying

1. abstractness: if $\phi \equiv_\Sigma \psi$ then $\prec_{\Sigma,\phi} = \prec_{\Sigma,\psi}$.
2. transitivity: $x \prec_{\Sigma,\phi} y \prec_{\Sigma,\phi} z$ implies $x \prec_{\Sigma,\phi} z$
3. base: if $x \Vdash_\Sigma \phi$ and $y \nVdash_\Sigma \phi$ then $x \prec_{\Sigma,\phi} y$; and if $x \prec_{\Sigma,\phi} y$ then $y \nVdash_\Sigma \phi$ and ϕ is satisfiable.
4. limit: there is no infinite descending chain of the form $\ldots x_3 \prec_{\Sigma,\phi} x_2 \prec_{\Sigma,\phi} x_1$.

3.0.11 Definition. A morphism h between preferential institutions $\langle \mathrm{Sig}, \mathrm{Sen}, \mathrm{Int}, \Vdash, \prec \rangle$ and $\langle \mathrm{Sig}', \mathrm{Sen}', \mathrm{Int}', \Vdash', \prec' \rangle$ is a morphism between the institutions $\langle \mathrm{Sig}, \mathrm{Sen}, \mathrm{Int}, \Vdash \rangle$ and $\langle \mathrm{Sig}', \mathrm{Sen}', \mathrm{Int}', \Vdash' \rangle$ such that, for all Σ in Sig, $x, y \in \mathrm{Int}(\Sigma)$ and $\phi' \in \mathrm{Sen}'(h(\Sigma))$:

$$x \prec_{\Sigma, h_{\alpha_\Sigma}(\phi')} y \text{ implies } h_{\beta_\Sigma}(x) \prec'_{h(\Sigma),\phi'} h_{\beta_\Sigma}(y)$$

However, the adjunctions described in this paper do not extend to the case of such default institutions without further work.

4. Conclusions

Starting from a class of interpretation structures $\mathcal{M}$, we have examined several proposals for structuring the class in order to equip it for reasoning with defaults. The two most intuitive proposals are $\mathcal{P}$, which is based on preference relations, and $\mathcal{D}$, which is based on distance functions. Various adjunctions were found, which relate the proposals in terms of granularities of defeasibility. These are summarised in figure 1.

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6. Appendix

Details of verifications too long for the main body of the paper are given here.

Proof of 2.3.3 (a) Let $(d', <')$ be $L_1(<)$; then $d'(x,u) = \{(z,w) \mid z,w \leqslant x,u\}$ and $d'(x,u) <' d'(y,v) \Leftrightarrow d'(x,u) \subset d'(y,v)$. Plainly $<'$ is a strict partial order. We must also verify the conditions of $\mathcal{D}$.

1. Base: Suppose $y \neq v$; we must show that $d'(x,x) <' d'(y,v)$. Suppose $(z,w) \in d'(x,x)$; then $z,w \leqslant x,x$. But $x,x < y,v$ by $\mathcal{D}^-$-base; so $z,w \leqslant y,v$ and $(z,w) \in d'(y,v)$. Hence $d'(x,x) \subseteq d'(y,v)$. To show strict subset, observe that $(y,v) \in d'(y,v)$ but, since $(y,v) \neq (x,x)$ and (x,x) is $<$-minimal, $(y,v) \notin d'(x,x)$.

   For the second part of base, suppose that $d'(x,u) <' d'(y,v)$; we must show that $y \neq v$. Suppose $y = v$; then $d'(y,v) = \{(y,v)\}$. However, $(x,u) \in d'(x,u) \subset d'(y,v)$ implies that $d'(y,v)$ is a proper superset of $\{(x,u)\}$, so contains at least two elements, a contradiction. Therefore, $y \neq v$.

2. Chain. Suppose $\ldots < d(x_3,u_3) < d(x_2,u_2) < d(x_1,u_1)$ is a chain. Then $\{(z,w) \mid z,w \leqslant x_3,u_3\} \subset \{(z,w) \mid z,w \leqslant x_2,u_2\} \subset \{(z,w) \mid z,w \leqslant x_1,u_1\}$, from which we can extract the chain $\ldots < x_3,u_3 < x_2,u_2 < x_1,u_1$, contradicting $\mathcal{D}^-$-chain.

(b) The conditions follow directly from their counterparts in $\mathcal{D}$.

(c) Suppose $<_1 \to <_2$; we prove $(d'_1, <'_1) \to (d'_2, <'_2)$, where $(d'_i, <'_i)$ is $L_1(<_i)$.

$$\begin{aligned} & d'_1(x,u) <'_1 d'_1(y,v) \\ \Leftrightarrow\ & \forall z,w.\ (z,w \leqslant_1 x,u \Rightarrow z,w \leqslant_1 y,v) \wedge \exists z,w.\ (z,w \not\leqslant_1 x,u \wedge z,w \leqslant_1 y,v) \\ \Leftrightarrow\ & x,u <_1 y,v \\ \Rightarrow\ & x,u <_2 y,v \\ \Leftrightarrow\ & d'_2(x,u) <'_2 d'_2(y,v). \end{aligned}$$

(d) Suppose $(d_1, <_1) \to (d_2, <_2)$; we prove $<'_1 \to <'_2$, where $<'_i$ is $R_1(d_i, <_i)$.

$$\begin{aligned} x,u <'_1 y,v \Leftrightarrow\ & d(x,u) <_1 d(y,v) \\ \Rightarrow\ & d(x,u) <_2 d(y,v) && \text{using hypothesis} \\ \Leftrightarrow\ & x,u <'_2 y,v \end{aligned}$$

Proof of 2.3.6 (a)

1. Weak transitivity: suppose $x, M\ L_2(\prec)\ y, M\ L_2(\prec)\ z, M$. Then $[x \prec_M y \vee (x \in M \wedge y \notin M)] \wedge [y \prec_M z \vee (y \in M \wedge z \notin M)]$. Since $x \in M \wedge y \notin M$ implies $x \prec_M y$, and similarly for the case with y and z, this condition reduces to $x \prec_M y \wedge y \prec_M z$. Therefore, by $\prec_M$-transitivity, $x \prec_M z$ and so $x, M\ L_2(\prec)\ z, M$.

   Turning to the second case in weak transitivity, suppose $x, \{u\}\ L_2(\prec)\ y, \{v\}\ L_2(\prec$ $z, \{w\}$. A similar case analysis to the one above, together with applications of $\mathcal{P}$-base and $\mathcal{P}$-transitivity, leads to the conclusion that $x, \{u\}\ L_2(\prec)\ z, \{w\}$.

2. Base: $x \in M \wedge y \notin N$ implies $x, M\ L_2(\prec)\ y, N$ trivially.

   For the second half, suppose $x, M\ L_2(\prec)\ y, N$, i.e. $(\exists K.\ N \subseteq K \subseteq M \wedge x \prec_K y) \vee (x \in M \wedge y \notin N)$. We show that each disjunct implies $y \notin N \wedge M \neq \varnothing$. For the second disjunct, it's immediate. For the first, pick such a K. By $\mathcal{P}$-base, $K \neq \varnothing$ and $y \notin K$. Since $N \subseteq K \subseteq M$, this means $M \neq \varnothing$ and $y \notin N$.

3. Chain: suppose $\ldots\ L_2(\prec)\ x_3, \{u_3\}\ L_2(\prec)\ x_2, \{u_2\}\ L_2(\prec)\ x_1, \{u_1\}$ was such a chain; we will obtain a contradiction by finding a chain that violates $\mathcal{P}$-chain. For each $i > 0$, we have $(u_{i+1} = u_i \wedge x_{i+1} \prec_{\{u_i\}} x_i) \vee (x_{i+1} = u_{i+1} \wedge x_i \neq u_i)$. We will show that the second disjunct cannot hold for any i. For suppose $x_{i+1} = u_{i+1}$; then taking $i+1$ in the formula stated, we obtain $(u_{i+2} = u_{i+1} \wedge x_{i+2} \prec_{\{u_{i+1}\}} x_{i+1}) \vee (x_{i+2} = u_{i+2} \wedge x_{i+1} \neq u_{i+1})$, and since the second disjunct cannot hold by hypothesis, we obtain $x_{i+2} \prec_{\{u_{i+1}\}} x_{i+1}$. But by $\mathcal{P}$-base, this implies $x_{i+1} \neq u_{i+1}$, contradicting the hypothesis. Therefore, we find that $\forall i > 0.\ (u_{i+1} = u_i \wedge x_{i+1} \prec_{\{u_i\}} x_i)$, giving us the chain $\ldots \prec_{\{u_1\}} x_3 \prec_{\{u_1\}} x_2 \prec_{\{u_1\}} x_1$ which violates $\mathcal{P}$-chain.

   Now suppose $\ldots\ L_2(\prec)\ x_3, M\ L_2(\prec)\ x_2, M\ L_2(\prec)\ x_1, M$; the argument is similar. We get $\forall i > 0.\ [x_{i+1} \prec_M x_i \vee (x_{i+1} \in M \wedge x_i \notin M)]$. Again, we argue that the second disjunct can never be selected; since it implies $x_{i+1} \in M$, and either disjunct of the condition for $i+1$ gives us $x_{i+1} \notin M$. We therefore must have $\forall i > 0.\ x_{i+1} \prec_M x_i$, contradicting $\mathcal{P}$-chain.

4. Unions: suppose $I \neq \varnothing$ and $x, M_i\ L_2(\prec)\ y, N_i$ for each $i \in I$. Then for each $i \in I$, $(\exists K_i.\ N_i \subseteq K_i \subseteq M_i \wedge x \prec_{K_i} y) \vee (x \in M_i \wedge y \notin N_i)$. Let $M = \bigcup_{i \in I} M_i$ and $N = \bigcup_{i \in I} N_i$. Note that $y \notin N_i$ for each i, since $x \prec_{K_i} y$ implies $y \notin K_i$ implies $y \notin N_i$. If for some i, $x \in M_i$, then $x \in M \wedge y \notin N$ and so $x, M\ L_2(\prec)\ y, N$. Otherwise, for each i, $\exists K_i.\ N_i \subseteq K_i \subseteq M_i \wedge x \prec_{K_i} y$. Put $K = \bigcup_{i \in I} K_i$; then $N \subseteq K \subseteq M$ and by $\mathcal{P}$-unions, $x \prec_K y$.

5. Monotonicity: clearly, if $(\exists K.\ N \subseteq K \subseteq M \wedge x \prec_K y) \vee (x \in M \wedge y \notin N)$ and $M \subseteq M'$ and $N' \subseteq N$ then $(\exists K.\ N' \subseteq K \subseteq M' \wedge x \prec_K y) \vee (x \in M' \wedge y \notin N')$.

(b) The properties follow directly from their corresponding properties in $\mathcal{DP}$; we will illustrate the case for $\mathcal{P}$-chain.

4. Chain: from the chain $\ldots x_3 \; R_2(\sqsubset)_M \; x_2 \; R_2(\sqsubset)_M \; x_1$ we obtain the chain $\ldots x_3, M \sqsubset x_2, M \sqsubset x_1, M$, violating $\mathcal{DP}$-chain.

(c)

1. Weak transitivity: we can verify the stronger condition of transitivity in this case.

$$\begin{aligned}
&x, M \; L_3(<) \; y, N \; L_3(<) \; z, K \\
&\quad \Leftrightarrow \; \forall k \in K.\, \exists n \in N.\, y, n < z, k \wedge \forall n \in N.\, \exists m \in M.\, x, m < y, n \\
&\qquad \wedge M \neq \varnothing \wedge N \neq \varnothing \\
&\quad \Rightarrow \; \forall k \in K.\, \exists n \in N.\, (y, n < z, k \wedge \exists m \in M.\, x, m < y, n) \wedge M \neq \varnothing \\
&\quad \Rightarrow \; x, M \; L_3(<) \; z, K
\end{aligned}$$

2. Base: suppose $x \in M \wedge y \notin N$; we need to prove $x, M \; L_3(<) \; y, N$, i.e. $\forall v \in N.\, \exists u \in M.\, x, u < y, v \wedge M \neq \varnothing$. Let $v \in N$. Then $v \neq y$. Put $u = x$. Then by $\mathcal{D}^-$-base, $x, u < y, v$. Moreover, $M \neq \varnothing$ since $x \in M$.

   Now suppose $x, M \; L_3(<) \; y, N$; we prove $y \notin N$ and $M \neq \varnothing$. The second part follows immediately when $L_3(<)$ is expanded. For the first part, suppose $y \in N$. Since $\forall v \in N.\, \exists u \in M.\, x, u < y, v$, by taking $v = y$ we obtain $x, u < y, y$ for some u, violating $\mathcal{D}^-$-base.

3. Chain: suppose $\ldots \; L_3(<) \; x_3, M_3 \; L_3(<) \; x_2, M_2 \; L_3(<) \; x_1, M_1$. $M_2 \neq \varnothing$ by base, so $\exists u_3 \in M_3.\, x_3, u_3 < x_2, u_2$. Therefore, $\exists u_4 \in M_4.\, x_4, u_4 < x_3, u_3$, etc, violating $\mathcal{D}^-$-chain.

4. Unions: suppose $I \neq \varnothing$ and $x, M_i \; L_3(\prec) \; y, N_i$ for each $i \in I$, i.e. $M_i \neq \varnothing \wedge \forall v \in N_i.\, \exists u \in M_i.\, x, y < y, v$. Set $M = \bigcup_{i \in I} M_i$ and $N = \bigcup_{i \in I} N_i$. Then $M \neq \varnothing$; and if $v \in N$, then $v \in N_i$ for some i, so pick $u \in M_i \subseteq M$ with $x, u < y, v$.

5. Monotonicity: Clearly $M \neq \varnothing \wedge \forall v \in N.\, \exists u \in M.\, x, u < y, v$ and $M \subseteq M'$ and $N' \subseteq N$ implies $M' \neq \varnothing \wedge \forall v \in N'.\, \exists u \in M'.\, x, u < y, v$.

(d) The properties follow directly from their corresponding properties in $\mathcal{DP}$; we will illustrate the case for $\mathcal{D}^-$-base.

3. Base: $y \neq u \;\; \Rightarrow \;\; x, \{x\} \sqsubset y, \{u\} \quad \mathcal{DP}\text{-base}$
   $\qquad\qquad\;\; \Rightarrow \;\; x, x \; R_3(\sqsubset) \; y, u.$

(e) Suppose $\prec \subseteq \prec'$.

$$\begin{aligned}
x, M \; L_2(\prec) \; y, N &\Leftrightarrow (\exists K.\, N \subseteq K \subseteq M \wedge x \prec_K y) \vee (x \in M \wedge y \notin N) \\
&\Rightarrow (\exists K.\, N \subseteq K \subseteq M \wedge x \prec'_K y) \vee (x \in M \wedge y \notin N) \\
&\Leftrightarrow x, M \; L_2(\prec') \; y, N
\end{aligned}$$

(f) Suppose $\sqsubset \subseteq \sqsubset'$.

$$\begin{aligned}
x \; R_2(\sqsubset)_M \; y &\Leftrightarrow x, M \sqsubset y, M \\
&\Rightarrow x, M \sqsubset' y, M \\
&\Leftrightarrow x \; R_2(\sqsubset')_M \; y
\end{aligned}$$

(g) Suppose $<_1 \subseteq <_2$.

$$\begin{aligned}
x, M \; L_3(<_1) \; y, N &\Leftrightarrow M \neq \varnothing \wedge \forall v \in N.\, \exists y \in M.\, x, u <_1 y, v \\
&\Rightarrow M \neq \varnothing \wedge \forall v \in N.\, \exists y \in M.\, x, u <_2 y, v \\
&\Leftrightarrow x, M \; L_3(<_2) \; y, N
\end{aligned}$$

(h) Suppose $\sqsubset_1 \subseteq \sqsubset_2$.

$$\begin{aligned}
x, u \; R_3(\sqsubset_1) \; y, v &\Leftrightarrow x, \{u\} \sqsubset_1 y, \{v\} \\
&\Rightarrow x, \{u\} \sqsubset_2 y, \{v\} \\
&\Leftrightarrow x, u \; R_3(\sqsubset_2) \; y, v
\end{aligned}$$

Proof of 2.3.7 We verify the following.

1. There is a morphism $L_2(R_2(\sqsubset)) \to \sqsubset$.

$$\begin{aligned}
x, M \; L_2(R_2(\sqsubset)) \; y, N &\Leftrightarrow (\exists K.\, N \subseteq K \subseteq M \wedge x \; R_2(\sqsubset)_K \; y) \vee (x \in M \wedge y \notin N) \\
&\Leftrightarrow (\exists K.\, N \subseteq K \subseteq M \wedge x, K \sqsubset y, K) \vee (x \in M \wedge y \notin N) \\
&\Rightarrow (N \subseteq M \wedge x, M \sqsubset y, N) \vee (x \in M \wedge y \notin N) \;\; \mathcal{DP}\text{-mono} \\
&\Rightarrow x, M \sqsubset y, N
\end{aligned}$$

2. There is an isomorphism $R_2(L_2(\prec)) \leftrightarrow \prec$.

$$\begin{aligned}
x \; R_2(L_2(\prec))_M \; y &\Leftrightarrow x, M \; L_2(\prec) \; y, M \\
&\Leftrightarrow x \prec_M y \vee (x \in M \wedge y \notin M) \\
&\Leftrightarrow x \prec_M y
\end{aligned}$$

3. There is a morphism $L_3(R_3(\sqsubset)) \to \sqsubset$.

$$\begin{aligned}
x, M \; L_3(R_3(\sqsubset)) \; y, N &\Leftrightarrow M \neq \varnothing \wedge \forall v \in N.\, \exists u \in M.\, x, u \; R_3(\sqsubset) \; y, v \\
&\Leftrightarrow M \neq \varnothing \wedge \forall v \in N.\, \exists u \in M.\, x, \{u\} \sqsubset y, \{v\} \\
&\Rightarrow \forall v \in N.\, x, \{u_v\} \sqsubset y, \{v\} && \text{skolemising} \\
&\Rightarrow x, \textstyle\bigcup_{v \in N} \{u_v\} \sqsubset y, N && \mathcal{DP}\text{-unions} \\
&\Rightarrow x, M \sqsubset y, N && \mathcal{DP}\text{-monotonicity}
\end{aligned}$$

4. There is an isomorphism $R_3(L_3(<)) \leftrightarrow <$.

$$\begin{aligned}
x, u \; R_3(L_3(<)) \; y, v &\Leftrightarrow x, \{u\} \; L_3(<) \; y, \{v\} \\
&\Leftrightarrow x, u < y, v.
\end{aligned}$$