Prioritising Preference Relations

Mark Ryan
Department of Computing, Imperial College
180 Queens Gate, London SW7 2BZ, United Kingdom
mdr@doc.ic.ac.uk

20 May 1993

Abstract

We describe some ideas and results about the following problem:

Given a set, a family of "preference relations" on the set, and a "priority" among those preference relations, which elements of the set are best? That is, which elements are most preferred by a consensus of the preference relations which takes account of their relative priority? The problem is posed in a deliberately general way, to capture a wide variety of examples.

Our main result gives sufficient conditions for the existence of 'best' elements for an important instance of the problem: preference relations are pre-orders, the priority among them is a partial order, and the definition of best elements uses a generalisation of lexicographic ordering.

1 Introduction

Often in computer science, we wish to pick from a set elements which satisfy certain properties. But often, the properties are conflicting, and we want to find elements which satisfy them as much as possible, or as many of them as possible. The situation may be complicated still further if some of the properties are more important than others.

Example 1 I like vegetarian food; I like nuts; I like tomatoes, especially with chili. My first preference is to take priority over the other two, which are incomparable in priority. Available are lamb casserole; nut roast with tomato sauce; and pasta with tomato chili sauce. Thus, there is no dish satisfying all of my preferences, but some are better at doing so than others.

Our motivation for studying this problem is to establish the foundations of *default reasoning.* We postpone discussion of this motivation until section 5.

For this paper, we assume that 'preferences' between elements in a set are expressed by a pre-order. We also stipulate that the priority between preferences is a partial order.

Our aim in this paper is to formalise the notion of prioritised preference, and to establish conditions under which 'best' (i.e. most preferred) elements may be found. Prioritised preference is formalised in section 2. Conditions for finding maximal elements are given in section 3. In section 4, we consider an alternative characterisation. Section 5 briefly describes applications of this work.

2 Prioritised preference

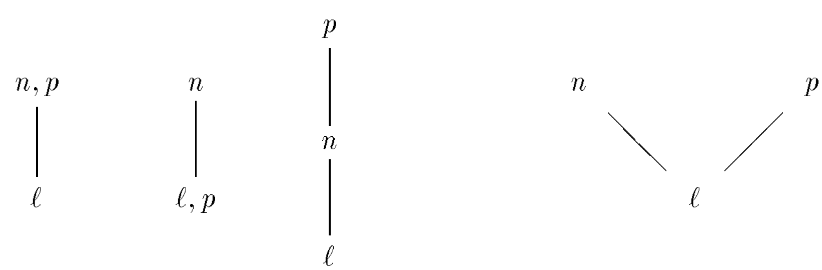
Definition 2 A pre-order is a reflexive and transitive relation. A partial order is a reflexive, anti-symmetric and transitive relation.

Notation 3 Let $(X, \sqsubseteq)$ be a pre-order or a partial order. We write $x \sqsubset y$ if $x \sqsubseteq y$ and $y \not\sqsubseteq x$; and $x \equiv y$ if $x \sqsubseteq y$ and $y \sqsubseteq x$.

Definition 4 Let X be a set. A *prioritised family of preferences* on X is a triple $(I, \leqslant, \{\sqsubseteq_i\}_{i \in I})$, where

- $(I, \leqslant)$ is a finite, partially ordered set, and
- for each $i \in I$, $\sqsubseteq_i$ is a pre-order on X.

The food preferences of example 1 may be represented as a family of prioritised preferences over the set of dishes $X = \{\ell, n, p\}$. In this case, $I = \{V, N, T\}$ (each letter representing a preference) with $< \; = \{(V, N), (V, T)\}$ and the orderings $\sqsubseteq_V$, $\sqsubseteq_N$, $\sqsubseteq_T$ are respectively the first three of



The fourth ordering represents the consensus of the prioritised preferences, which we will shortly define.

Notice that $i < j$ means that $\sqsubseteq_i$ has a *higher* priority than $\sqsubseteq_j$. But $x \sqsubset_i y$ means that y is preferred to x by i.

Definition 5 Let $(I, \leqslant, \{\sqsubseteq_i\}_{i \in I})$ be a prioritised family of preferences on a set X. The *globalisation* or *consensus* of $\{\sqsubseteq_i\}_{i \in I}$ is the relation $\sqsubseteq$ on X defined by

$$x \sqsubseteq y \quad \text{if} \quad \forall i \in I.\,(x \sqsubseteq_i y \text{ or } \exists j \in I.\,(j < i \text{ and } x \sqsubset_j y)).$$

That is, y is as preferred as x overall if it is as preferred according to each of the relations except possibly those for which there is a relation of greater priority, at which y is strictly preferred to x.

The remainder of this section is devoted to theorems and remarks which further motivate and explain this definition.

Lemma 6 $x \sqsubseteq y$ iff $\forall i \in I.\,(x \sqsubseteq_i y \text{ or } (\exists j < i.\ x \sqsubset_j y \text{ and } \forall k < j.\ x \sqsubseteq_k y))$.

Proof $\Leftarrow$ immediate. $\Rightarrow$ Find j minimal with $x \sqsubset_j y$. □

The definition of globalisation is a generalisation to partial orders of the usual lexicographic ordering. That is to say, if $(I, \leqslant)$ is a total ordering, say $\{1 < 2 < \cdots < n\}$ then the globalisation is the following:

$$\begin{array}{lll} x \sqsubseteq y & \text{if} & x \sqsubset_1 y \\ & \text{or} & x \sqsubseteq_1 y \text{ and } x \sqsubset_2 y \\ & \text{or} & x \sqsubseteq_1 y,\ x \sqsubseteq_2 y \text{ and } x \sqsubset_3 y \\ & \text{or} & \cdots \\ & \text{or} & x \sqsubseteq_1 y,\ x \sqsubseteq_2 y\ \ldots \text{ and } x \sqsubseteq_n y \end{array}$$

Our convention that $i < j$ implies $\sqsubseteq_i$ has greater priority than $\sqsubseteq_j$ matches with the lexicographic ordering.

Proposition 7 The globalisation $\sqsubseteq$ of a prioritised family of preference relations is a pre-order.

Proof Reflexivity is obvious. For transitivity, suppose $x \sqsubseteq y \sqsubseteq z$, and let $i \in I$. We shall show $x \sqsubseteq_i z$ or $x \sqsubset_j z$ for some $j < i$.

Suppose $x \sqsubseteq_i y$. If $y \sqsubseteq_i z$ then $x \sqsubseteq_i z$. Otherwise, $y \not\sqsubseteq_i z$, so let $j_1 < i$ be such that $y \sqsubset_{j_1} z$ and $y \sqsubseteq_k z$ for $k < j_1$ (lemma 6). If $x \not\sqsubseteq_{j_1} y$, then let

$j < j_1$ be such that $x \sqsubset_j y$. Then $j < i$ and $x \sqsubset_j z$ follows from $x \sqsubset_j y$ and $y \sqsubseteq_j z$. If $x \sqsubseteq_{j_1} y$, set $j = j_1$. Then $j < i$, and $x \sqsubset_j z$ follows from $x \sqsubseteq_j y$ and $y \sqsubset_j z$.

On the other hand, suppose $x \not\sqsubseteq_i y$ and let $j_2 < i$ be such that $x \sqsubset_{j_2} y$ and $x \sqsubseteq_k y$ for all $k < j_2$ (lemma 6). Again, consider separately the two cases $y \sqsubseteq_{j_2} z$ and $y \not\sqsubseteq_{j_2} z$. If $y \sqsubseteq_{j_2} z$, set $j = j_2$; then $j < i$, and $x \sqsubset_j z$ follows from $x \sqsubset_j y$ and $y \sqsubseteq_j z$. Otherwise, $y \not\sqsubseteq_{j_2} z$ so let $j < j_2$ be such that $y \sqsubset_j z$; then $j < i$, and $x \sqsubset_j z$ follows from $x \sqsubseteq_j y$ and $y \sqsubset_j z$. □

Definition 5 has also been described in various guises in Ryan [5] and Grosof [3].

3 Finding maximal elements

We are interested in finding elements which are 'best' according to the consensus of the prioritised preference relations; that is, we are interested in finding $\sqsubseteq$-maximal elements. As is well-known, **Zorn's lemma** [1] can be used to prove that maximal elements of an ordering exist. Zorn's lemma says that a pre-order has a maximal element if every chain in the pre-order has an upper bound.

Definition 8

1. Let $(X, \sqsubseteq)$ be a pre-order. A subset Y of X is a $\sqsubseteq$-*chain* if Y is totally ordered by $\sqsubseteq$; that is, $\forall y, z \in Y.\ y \sqsubseteq z$ or $z \sqsubseteq y$.

2. A chain Y in $(X, \sqsubseteq)$ has an *upper bound* $a \in X$ if $\forall y \in Y.\ y \sqsubseteq a$.

Proposition 9 (Zorn's Lemma) A pre-order $(X, \sqsubseteq)$ has a maximal element if every chain in X has an upper bound.

Before we apply Zorn's lemma, we establish the existence of certain key preferences which will enable us to reduce $\sqsubseteq$-chains to $\sqsubseteq_i$-chains. We do this in the first of the following two lemmas. Then, in the second of the two, we compose the $\sqsubseteq_i$ chains to give another chain. We show that this chain has an upper bound, and that its upper bound serves as an upper bound of the $\sqsubseteq$ chain we started with.

The next few definitions and lemmas are technical lemmas whose real purpose is to assist in the proof of theorem 15.

Definition 10 Let $(I, \leqslant, \{\sqsubseteq_i\}_{i \in I})$ be a prioritised family of preferences on a set X, and let $x, y \in X$. The x, y-*frontier*, written $\mathrm{fr}(x, y)$, is the set of $\leqslant$-minimal elements of the set $\{i \in I \mid x \not\equiv_i y\}$.

Note that if $\{i \in I \mid x \not\equiv_i y\} = \varnothing$ then $\mathrm{fr}(x, y) = \varnothing$.

Lemma 11 Suppose $x \sqsubseteq y$. Then $i \in \mathrm{fr}(x, y)$ iff $x \sqsubset_i y$ and $\forall j < i.\ x \equiv_j y$.

Proof (If) Immediate. (Only if) Let $x \sqsubseteq y$ and $i \in \mathrm{fr}(x, y)$. (1) We prove $x \sqsubseteq_i y$; for if not, by definition 5 $\exists j < i.\ x \sqsubset_j y$, i.e. $x \not\equiv_j y$, contradicting i's minimality. (2) Since $i \in \mathrm{fr}(x, y)$, $x \not\equiv_i y$. Thus $x \sqsubset_i y$.

Now suppose $j < i$. Since i is minimal in $\{i \in I \mid x \not\equiv_i y\}$, we have $x \equiv_j y$. □

Definition 12 Let $J \subseteq I$. We write $x \sqsubseteq_J y$ if $\forall j \in J.\ x \sqsubseteq_j y$. We also write $\downarrow J$ for $\{i \in I \mid \exists j \in J.\ i \leqslant j\}$.

Lemma 13 Let $Y \subseteq X$ be a $\sqsubseteq$-chain with no maximal element. Then there exists $J \subseteq I$ and $a \in Y$ such that

1. $\forall j \in J.\ \forall i \in I.\ \forall x, y \in Y.\ (a \sqsubseteq x \sqsubseteq y$ and $i \leqslant j)$ implies $x \sqsubseteq_i y$ — that is, $\{y \in Y \mid a \sqsubseteq y\}$ forms a $\sqsubseteq_{\downarrow J}$-chain.

2. $\forall j \in J.\ \forall x \in Y.\ a \sqsubseteq x$ implies $\exists z \in Y.\ (x \sqsubset z$ and $x \sqsubset_j z)$ — that is, the same set also forms a $\sqsubseteq_J$-chain with no maximal element.

3. $\forall i \in I.\ \forall x, y \in Y.\ a \sqsubseteq x \sqsubseteq y$ implies $(x \sqsubseteq_i y$ or $\exists j \in J.\ j < i)$.

Proof The idea of the proof is the following. First, we obtain a set $I' \subseteq I$ which contains those i which participate in frontiers all the way up the chain Y. Then find an element a of Y above which *all* the frontiers are in I'. J is defined as the minimal elements of I'. Then it is possible to prove property 1. Property 2 follows because we have stipulated that Y have no maximal element; that is, for each $y \in Y$ there is a $y' \in Y$ with $y \sqsubset y'$. Property 3 follows because J is the set of minimal elements of I'.

Let $I' = \{i \in I \mid \forall x \in Y.\ \exists y, z \in Y.\ x \sqsubseteq y \sqsubset z$ and $i \in \mathrm{fr}(y, z)\}$.

- If $I' = I$ then let a be an arbitrary element of Y.

- Otherwise, for each $i \in I - I'$ let $x_i \in Y$ be such that $\forall y, z \in Y$, if $x_i \sqsubseteq y \sqsubset z$ then $i \notin \mathrm{fr}(y, z)$, and let $a = \max_{\sqsubseteq}\{x_i \mid i \in I - I'\}$. That each x_i can be found follows from the definition of I', and that their maximum can be found is guaranteed by the facts that Y is a chain and I is finite.

Now we show that I' is non-empty. Let $x, y \in Y$ be such that $a \sqsubseteq x \sqsubset y$. The fact that Y has no maximal element guarantees that these can be found. Since $x \sqsubset y$, $\mathrm{fr}(x, y) \neq \emptyset$, and since $a \sqsubseteq x, y$, we have $\mathrm{fr}(x, y) \subseteq I'$.

1. Let $j \in J$, $i \in I$ and $x, y \in Y$ be such that $i \leqslant j$ and $a \sqsubseteq x \sqsubseteq y$. If $i \in \mathrm{fr}(x, y)$ then $x \sqsubset_i y$ (lemma 11); otherwise, if $i \notin \mathrm{fr}(x, y)$ and $x \not\sqsubseteq_i y$ then $\exists j' < i.\ x \sqsubset_{j'} y$, contradicting the minimality of j in J.

2. Let $j \in J$ and $x \in Y$ with $a \sqsubseteq x$. Since $j \in I'$, we can pick $y, z \in Y$ with $x \sqsubseteq y \sqsubset z$ and $j \in \mathrm{fr}(y, z)$. By part 1, $x \sqsubseteq_j y \sqsubseteq_j z$; and since $j \in \mathrm{fr}(y, z)$ we have $y \sqsubset_j z$. By transitivity, $x \sqsubset_j z$.

3. If $x \not\sqsubseteq_i y$ then $\exists j' \in \mathrm{fr}(x, y) \subseteq I'.\ j' < i$ (lemma 6), and since J consists of the minimal elements of I' (and I is finite!), $\exists j \in J.\ j < j'$. □

Now we show, subject to a certain condition, that it is possible to find an upper bound for any $\sqsubseteq$-chain. The condition says that upper bounds can be found for intersections (i.e. conjunctions) of the $\sqsubseteq_i$ relations.

Lemma 14 Suppose for every $J \subseteq I$, every $\sqsubseteq_J$-chain has an upper bound. Then every $\sqsubseteq$-chain has an upper bound.

Proof Let Y be a $\sqsubseteq$-chain. If Y has a maximal element, then that serves as its upper bound. Suppose, then, that Y has no maximal element. Let $J \subseteq I$ and $a \in Y$ be as defined in lemma 13. Let $K = J \cup \{k \in I \mid \forall j \in J.\ j \nleqslant k\}$. We now show that the set $\{x \in Y \mid a \sqsubseteq x\}$ forms a $\sqsubseteq_{\downarrow K}$ chain. Without loss of generality, let $x, y \in Y$ be such that $a \sqsubseteq x \sqsubseteq y$, and $i \in I$ and $k \in K$ be such that $i \leqslant k$. We need to show that $x \sqsubseteq_i y$. If $k \in J$ then $x \sqsubseteq_i y$ by lemma 13(1). Otherwise, $\forall j \in J.\ j \nleqslant k$ (definition of K). Therefore, $j \nleqslant i$. Suppose $x \not\sqsubseteq_i y$. Then by 13(3), $\exists j \in J.\ j < i$, a contradiction. So $x \sqsubseteq_i y$.

Now let b be a $\sqsubseteq_{\downarrow K}$ upper bound for $\{x \in Y \mid a \sqsubseteq x\}$. We show that it is also a $\sqsubseteq$ upper bound for that set, and hence for Y. Let $x \in Y$ with $a \sqsubseteq x$; we show that $x \sqsubseteq b$, using definition 5.

First note that (i) $j \in {\downarrow K}$ implies $x \sqsubseteq_j b$ (by definition of b). Also, (ii) $j \in J$ implies $x \sqsubset_j b$. To see this, take y such that $x \sqsubset_j y$ by lemma 13(2); but then $y \sqsubseteq_j b$, so $x \sqsubset_j b$.

Now let $i \in I$. We show that either $x \sqsubseteq_i b$ or $\exists j < i.\ x \sqsubset_j b$. If $i \in {\downarrow K}$, $x \sqsubseteq_i b$ by (i). If $i \notin {\downarrow K}$, then $i \notin K$. By definition of K, $\exists j \in J.\ j < i$; by (ii), $x \sqsubset_j b$. □

Thus, we are in a position to provide sufficient conditions for being able to find maximal elements.

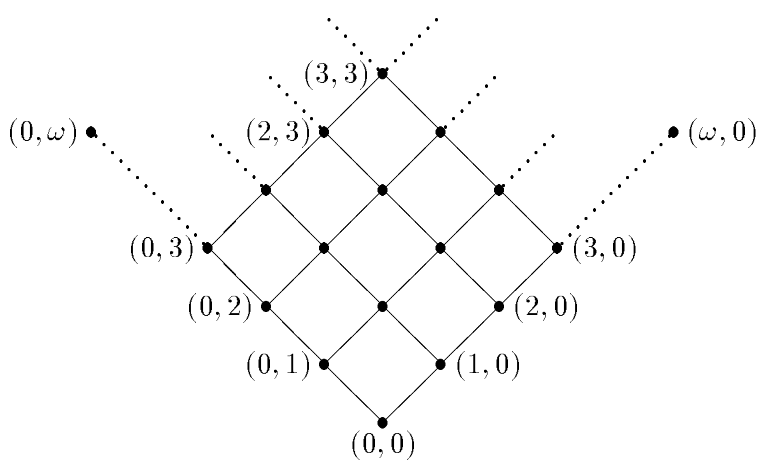
Theorem 15 Let $(I, \leqslant, \{\sqsubseteq_i\}_{i\in I})$ be a prioritised family of preferences on a set X with globalisation $\sqsubseteq$, such that for every $J \subseteq I$, every $\sqsubseteq_J$-chain has an upper bound. Then $\sqsubseteq$ has maximal elements.

Proof By Zorn's lemma and lemma 14. □

The sufficient conditions for finding maximal elements may feel a bit unsatisfactory, and one might ask whether they can be weakened. For example, maybe being able to find upper bounds on all $\sqsubseteq_i$-chains (but not necessarily their intersections) is sufficient. However, the following example shows that this is not so.

Example 16 Let $X = (\mathbb{N} \times \mathbb{N}) \cup \{(0, \omega), (\omega, 0)\}$. Let $I = \{1, 2\}$ with $\leqslant$ the discrete ordering (i.e. $< \, = \varnothing$) and

$$\begin{array}{llll} (x,y) \sqsubseteq_1 (x',y') & \text{if} & x \leqslant x' & \text{according to the numerical ordering} \\ (x,y) \sqsubseteq_2 (x',y') & \text{if} & y \leqslant y' & \text{ditto} \end{array}$$



Notice that although the premise of lemma 14 fails, all $\sqsubseteq_1$-chains and $\sqsubseteq_2$-chains have upper bounds. The globalisation is defined as $(x, y) \sqsubseteq (x', y')$ if $x \leqslant x'$ and $y \leqslant y'$. It has no maximal element. The chain $(0, 0) \sqsubseteq (1, 1) \sqsubseteq (2, 2) \sqsubseteq \cdots$ has no upper bound according to this relation. So this example shows that upper bounds on the $\sqsubseteq_i$ orderings is not enough.

4 Are these really the best elements?

The contribution of the preceding two sections is as follows: in the first section, we gave the definition of the 'globalising' or 'consensus' relation for a prioritised family of preferences, and examined its properties. In the second section, we showed that (under certain conditions) it is possible to find 'best' elements according to those preferences – namely, those maximal in the globalising relation.

Are these really the best elements? There is a particular case of prioritised preference relations in which one may have conflicting intuitions. In this section we examine those intuitions and propose an alternative definition.

4.1 Totally prioritised families of preferences

An order $\leqslant$ is *total* if for each $i, j \in I$, we have $i \leqslant j$ or $j \leqslant i$.

Suppose our $(I, \leqslant)$ is a total order $\{1 < 2 < \cdots < n\}$. We will write this *totally prioritised family of preferences* as $\{\sqsubseteq_i\}_{i \leqslant n}$. Note that the preference orderings $\sqsubseteq_i$ need not be total; it is the priority relation $\leqslant$ which is total.

This notation is convenient because in this case we can characterise the globalising relation inductively. We write the globalisation of $\{\sqsubseteq_i\}_{i \leqslant n}$ as $\sqsubseteq^n$.

Proposition 17

1. $x \sqsubseteq^0 y$ always; and
2. $x \sqsubseteq^i y$ if $x \sqsubset_i y$ or ($x \equiv_i y$ and $x \sqsubseteq^{i-1} y$).

The proof is straightforward. In this case we may expect that the maximal elements in the globalisation relation $\sqsubseteq^n$ can be found in the following way.

Procedure 18 (incorrect) Start with the set X. Take the maximal elements according to the relation $\sqsubseteq_1$ (it has the highest priority). Then, from the resulting set, choose those which are maximal in $\sqsubseteq_2$. From that set, take those which are maximal according to $\sqsubseteq_3$. Continue in this way until each of the relations has been considered.

Any element found in this way is indeed $\sqsubseteq^n$-maximal, but it turns out that this does not yield *all* of the $\sqsubseteq^n$ maximal elements. To find all the maximal elements, proceed as follows.

Procedure 19 (correct) Start, as before, with the set X. Take the $\sqsubseteq_1$-maximal equivalence classes. This is a set of sets; each element inside the sets is $\sqsubseteq_1$-maximal, and each set is a $\equiv_1$ equivalence class. Now for each such equivalence class, take the $\sqsubseteq_2$-maximal equivalence classes. This gives another set of sets, several of them resulting from each one of the previous set of sets. Continue in this way until all the relations have been considered.

To formalise this procedure and state the necessary theorem, we need the following notation.

Definition 20 Let $(X, \sqsubseteq)$ be a pre-order, and $\equiv$ the corresponding equivalence relation. If $Y \subseteq X$, then $\mathrm{Max}_{\sqsubseteq}(Y)$ is the set of $\equiv|_Y$-equivalence classes which are $\sqsubseteq$-maximal. That is,

$$\begin{aligned} \mathrm{Max}_{\sqsubseteq}(Y) = \{Z \subseteq Y \mid \forall z \in Z.\ (&z \text{ is } \sqsubseteq\text{-maximal in } Y, \text{ and} \\ &\forall x \in X.\ x \in Z \text{ iff } x \equiv z)\}. \end{aligned}$$

Notice that this is not the set of maximal elements of Y; rather, it is a set of sets whose union is the set of maximal elements. It is, in fact, the set of maximal elements of Y partitioned by $\equiv|_Y$.

Proposition 21 Let $\{\sqsubseteq_i\}_{i \leqslant n}$ be a totally prioritised family of preferences on X. Define the sets $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n \subseteq \mathcal{P}(X)$ as follows.

1. $\mathbf{X}_0 = \{X\}$, and
2. $\mathbf{X}_j = \{Y \subseteq X \mid \exists Z \in \mathbf{X}_{j-1}.\ Y \in \mathrm{Max}_{\sqsubseteq_j}(Z)\}$ $(1 \leqslant j \leqslant n)$.
   (Equivalently, $\mathbf{X}_j = \bigcup_{Z \in \mathbf{X}_{j-1}} \mathrm{Max}_{\sqsubseteq_j}(Z)$.)

Then: x is $\sqsubseteq^n$-maximal iff $\exists Y \in \mathbf{X}_n.\ x \in Y$.

In spite of this rather awkward notation, the difference between the two procedures is easy to see.

Example 22 Suppose $X = \{a, b\}$, $\sqsubseteq_1 = \{(a, a), (b, b)\}$, and $\sqsubseteq_2 = \{(a, a), (b, b), (a, b)\}$. That is, $\sqsubseteq_1$ and $\sqsubseteq_2$ are

Procedures 18 and 19 yield $\{b\}$ and $\{a, b\}$ respectively as the set of $\sqsubseteq^2$-maximal elements.

Procedure 18 takes the view that as $\sqsubseteq_1$ has not decided the matter between a and b, then it should be up to $\sqsubseteq_2$ to determine that b is superior to a and thus that b is overall maximal. On the other hand, procedure 19 takes the view that $\sqsubseteq_1$ has decided the matter between a and b; it says that they are incomparable. Since $\sqsubseteq_2$ is of less priority, it gets no say.

Example 23 Suppose $X = \{a, b\}$, $\sqsubseteq_1 = \{(a,a),(b,b),(a,b),(b,a)\}$, and $\sqsubseteq_2 = \{(a,a),(b,b),(a,b)\}$. That is, $\sqsubseteq_1$ and $\sqsubseteq_2$ are

Both procedures yield $\{b\}$ as the set of $\sqsubseteq^2$-maximal elements.

Moral of story: there is a difference between saying that two elements are incomparable and saying that they are equivalent!

4.2 Squashing pre-orders into total orders

Our final question is this: is there a way of defining the globalisation of a family of prioritised relations in such a way that, in the totally prioritised case, procedure 18 yields the right results? The answer is 'yes', in certain conditions which we will state; but the definition is rather unnatural. We take this to be evidence of the unnaturalness of that procedure.

The essence of the definition is to squash the pre-orders $\sqsubseteq_i$ into total pre-orders $\sqsubseteq_i^*$ in such a way as to preserve maximal elements. Then we use the definition of globalisation (definition 5).

If $\sqsubseteq$ is a pre-order, $\sqsubseteq^*$ is the squashed version.

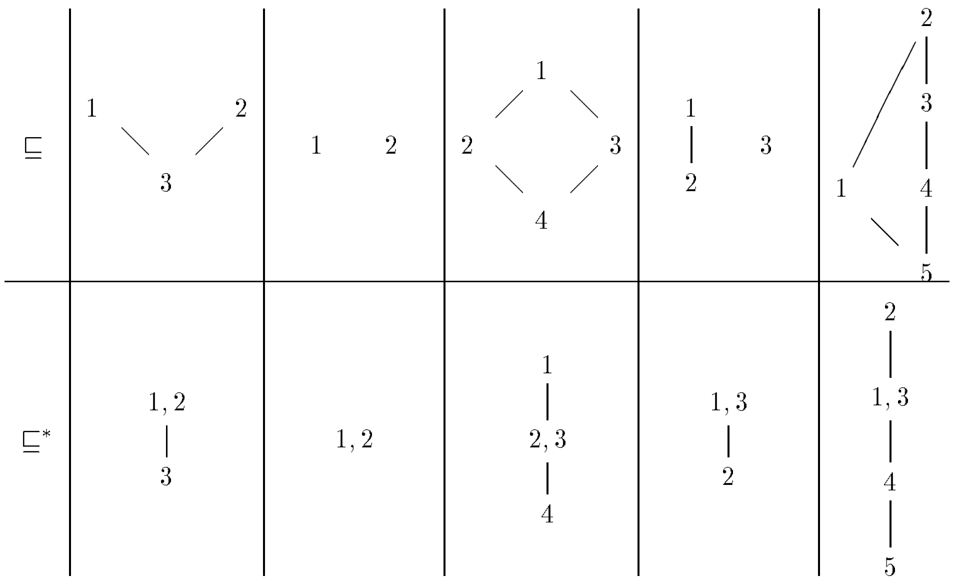
Definition 24 Let $(X, \sqsubseteq)$ be a pre-order. The pre-order $\sqsubseteq^*$ on X is defined as follows.

$$x \sqsubseteq^* y \quad \text{if} \quad |{\uparrow}x| \geqslant |{\uparrow}y|$$

where

- ${\uparrow}x = \{y \in X \mid x \sqsubset y\}$, and
- $|Y|$ is the cardinality of Y, and $\geqslant$ compares cardinalities.

Example 25 Here are some examples of $\sqsubseteq$ with the squashed version $\sqsubseteq^*$



Proposition 26

1. $\sqsubseteq^*$ is total.

2. If x is $\sqsubseteq$-maximal then x is $\sqsubseteq^*$-maximal.

3. If $\sqsubseteq$ has maximal elements then: if x is $\sqsubseteq^*$-maximal then x is $\sqsubseteq$-maximal.

Proof 2. x is $\sqsubseteq$-maximal implies $|{\uparrow}x| = 0$ implies x is $\sqsubseteq^*$-maximal.

3. Let y be any maximal element. Then $|{\uparrow}y| = 0$. Now suppose x is $\sqsubseteq^*$-maximal; then $|{\uparrow}x| = 0$, so x is $\sqsubseteq$-maximal. □

Proposition 27 Let $\{\sqsubseteq_i\}_{i \leqslant n}$ be a totally prioritised family of *total* preferences on X. (That is to say, each $\sqsubseteq_i$ is also total; $\forall x, y.\ x \sqsubseteq_i y$ or $y \sqsubseteq_i x$.) Then the globalisation $\sqsubseteq$ is also total.

Proof Suppose $y \not\sqsubseteq x$. We show that $x \sqsubseteq y$. Since $y \not\sqsubseteq x$, there is i such that $y \not\sqsubseteq_i x$ and $\forall k < i.\ y \not\sqsubset_k x$. But since these are total orders, this implies $x \sqsubset_i y$ and $\forall k < i.\ x \sqsubseteq_k y$. Therefore, $x \sqsubseteq y$. □

Proposition 28 Let $\{\sqsubseteq_i\}_{i \leqslant n}$ be a totally prioritised family of preferences on X, such that each $\sqsubseteq_i$ has maximal elements. Let $\sqsubseteq$ be the globalisation of $\{\sqsubseteq_i^*\}_{i \leqslant n}$ (that is, the squashed relations). Then x is $\sqsubseteq$-maximal iff x is obtained by procedure 18.

5 Applications

The motivation for studying this problem is to establish the foundations of *default reasoning*. In the framework we propose (which is the subject of [6]), defaults are represented by sentences in the language. The idea of default reasoning is that it is in general not possible to satisfy the defaults, but nevertheless we want models of our theory which are as close to satisfying the defaults as possible. To that end, we define orderings which measure how well an interpretation of a logical language satisfies a given sentence in the language. Thus, we write $M \sqsubseteq_\phi N$ to mean that N satisfies ϕ at least as well as M does. This definition and examples can be found in [6] and [5]. Clearly, we will be interested in interpretations which are $\sqsubseteq_\phi$-maximal.

Thus, a default denotes a preference among models. The techniques of this paper are then applied to putting defaults together with priorities. We thus give semantics to 'ordered theory presentations' (OTPs). An OTP is a partially ordered multiset of sentences, the ordering representing a priority on the sentences. The sentences are in general contradictory, and a model of an OTP is defined to be an interpretation of the language which is maximal in the globalised relation which comes from the individual preference relations in the way described in this paper.

This work also represents a direct generalisation of the notion of 'prioritised circumscription' [4] to the case that 'priority' is a partial order. In circumscription, minimizing a predicate means preferring interpretations which have a small extension of the predicate. The definition of (totally) prioritised circumscription uses, in effect, the lexicographic ordering described here.

Our main interest is to apply these techniques to reasoning about 'normative specifications' – that is, specifications which contain constraints which one has to maximise or minimise. Obviously, we are interested in the case where these constraints are ordered by priority. See [2, 7].

There are also applications to optimisation under (prioritised) constraints and in 'social choice theory' – but they are still to be investigated.

Acknowledgments

Many thanks to David Makinson and Pierre-Yves Schobbens for useful correspondence and discussion on early versions of these results. Thanks to the referees Reinhold Heckman, Roy Crole and Abbas Edalat for useful comments on a draft of this paper.

References

[1] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks. Cambridge University Press, 1990.

[2] J. Fiadeiro and T. Maibaum. Temporal reasoning over deontic specifications. *Journal of Logic and Computation*, 1(3):357–395, 1991.

[3] B. N. Grosof. Generalising prioritization. In J. Allen, R. Fikes, and E. Sandewall, editors, *Proc. Second International Conference on Principles of Knowledge Representation and Reasoning (KR'91)*, pages 289–300. Morgan Kaufmann, 1991.

[4] V. Lifschitz. Pointwise circumscription. In M. L. Ginsberg, editor, *Readings in Non-monotonic Logic*. Morgan Kaufmann, 1987.

[5] M. D. Ryan. Defaults and revision in structured theories. In *Proc. Sixth IEEE Symposium on Logic in Computer Science (LICS)*, pages 362–373, 1991.

[6] M. D. Ryan. *Ordered Presentations of Theories: Default Reasoning and Belief Revision*. PhD thesis, Department of Computing, Imperial College, 1992.

[7] M. D. Ryan. Towards specifying norms. *Annals of Mathematics and Artificial Intelligence*, 1993. In print.